Sufficient condition for a finite-time singularity in a high-symmetry Euler flow: Analysis and statistics

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A sufficient condition is obtained for the development of a finite-time singularity in a highly symmetric Euler flow, first proposed by Kida [J. Phys. Soc. Jpn. **54**, 2132 (1995)] and recently simulated by Boratav and Pelz [Phys. Fluids **6**, 2757 (1994)]. It is shown that if the second-order spatial derivative of the pressure (p_{xx}) is positive following a Lagrangian element (on the x axis), then a finite-time singularity must occur. Under some assumptions, this Lagrangian sufficient condition can be reduced to an Eulerian sufficient condition which requires that the fourth-order spatial derivative of the pressure (p_{xxxx}) at the origin be positive for all times leading up to the singularity. Analytical as well as direct numerical evaluation over a large ensemble of initial conditions demonstrate that for fixed total energy, p_{xxxx} is predominantly positive with the average value growing with the numbers of modes. [S1063-651X(96)13008-7]

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The mechanisms by which a fluid generates intense small-scale dynamics are crucial to our understanding of turbulence. Once small scales are created spontaneously, dissipation intervenes, and the dynamical balance between the two processes determines the character of turbulence. In particular, the dissipation rate of Navier-Stokes turbulence depends crucially on how the vorticity scales with the Reynolds number. Therefore, it is of great importance to study how small scales can be generated in a fluid by the action of vortex stretching controlled by the nonlinearities in the three-dimensional (3D) Euler equation (which is the infinite-Reynolds-number limit of the Navier-Stokes equation),

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p. \tag{1}$$

Here, for incompressible velocity fields, the self-consistent pressure p must satisfy the equation $\nabla^2 p=-\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})$. The main question is whether the solution to Eq. (1) becomes singular in finite time for a smooth initial condition with finite energy.

Mathematicians have provided some useful and rigorous constraints on the nature of possible singularities in 3D flows [1–4], but a physical model which explicitly demonstrates the singularity in a mathematically rigorous way remains elusive. It has been claimed [5,6] that a recent analytical model developed for a symmetric initial condition exhibits a finite-time singularity, but the demonstration relies on some strong assumptions which, while physically plausible, have not yet been substantiated formally or verified by a suitably designed numerical experiment. The analytical results [5] and [6] are suggestive: if one begins from an initial state with symmetries that are preserved by the Euler equation for all time, then the problem of finite-time singularities of the Euler equation could be somewhat more tractable.

This paper is stimulated by the recent numerical experiment of Boratav and Pelz (BP) [7] on a highly symmetric initial flow field, first proposed by Kida [8]. Due to the high symmetry of the Kida flow, BP were able to simulate the 3D Navier-Stokes equation with high spatial resolution and relatively small viscosity. (For the computer runs reported by BP, the Reynolds number Re $(=1/\nu)$ varies from 1000 to 5000 and the maximum total resolution is 1024^3.) Within the limits of the spatial resolution, BP report that the maximum vorticity scales as $(t_c-t)^{-1}$, and attribute its eventual saturation to the presence of viscosity. In a subsequent paper [9], BP report a loss of regularity in the strain tensor, and find, furthermore, that the spatial locations of the almost-divergent strain and vorticity are not coincident.

The principal goal of this paper is to present a sufficient condition for the development of a finite-time singularity in the Kida flow. The demonstration of this sufficient condition provides physical insight into a possible mechanism for singularity formation in this highly symmetric geometry. Two forms of this sufficient condition are given: a Lagrangian form for a moving point, and a more useful Eulerian form for a stationary point (the origin) that can be derived from the Lagrangian form under some assumptions. Though we are unable to provide an analytical proof, there is some numerical evidence [10] that the Eulerian sufficient condition is satisfied for the specific initial condition used in the numerical experiment of BP. We present additional statistical evidence that the underlying symmetries of the Kida flow make it highly probable that this condition is also valid for a large ensemble of initial conditions.

The symmetries of Kida flows have been discussed in detail in [8]. Here we build these symmetries into the representations for $\mathbf{v}$ and p. The components of the velocity field $\mathbf{v}=(v_x,v_y,v_z)$ can be written as $v_x=u(x,y,z)$, $v_y=u(y,z,x)$, $v_z=u(z,x,y)$, where u can be expressed in Fourier series,

$$u(x,y,z)=\sum_{lmn} a_{lmn}\sin lx\,\cos my\,\cos nz. \tag{2}$$

Here (l,m,n) are natural numbers which represent the three components of a wave vector $(l\neq 0)$. In order to satisfy the symmetries and the condition $\nabla\cdot\mathbf{v}=0$, the following conditions must hold:

$$l,m,n \text{ must be all odd or all even,} \tag{3}$$

$$a_{lmn}=(-1)^l a_{lnm}, \tag{4}$$

$$\sum_{lmn}^{C} l a_{lmn}=0, \tag{5}$$

where the last summation (denoted by C) is over all permutations of any three natural numbers (l,m,n), i.e., $la_{lmn}+ma_{mnl}+na_{nlm}=0$. By (2) and (5), it can be seen that for $\mathbf{x}$ close to the origin, $\mathbf{v}=O(|\mathbf{x}|^3)$. In particular, the initial state considered by both Kida [8] and BP is u_0: $a_{1,3,1}=1$, $a_{1,1,3}=-1$, with all other terms set to zero. For this initial state, $v_x=\partial v_x/\partial x=0$ at $t=0$ for all x.

With u represented by (2), it can be shown that the pressure p is of the form $p=\Sigma_{lmn}p_{lmn}\cos lx \cos my \cos nz$, where p_{lmn}, using the Poisson equation for the pressure, is given by

$$p_{lmn}=(lA_{lmn}+mA_{mnl}+nA_{nlm})/(l^2+m^2+n^2).$$

with A_{lmn} defined by

$$(\mathbf{v}\cdot\nabla\mathbf{v})_x\equiv\sum_{lmn} A_{lmn}\sin lx \cos my \cos nz.$$

It can be shown that A_{lmn} also satisfies (3) and (4) (with a_{lmn} replaced by A_{lmn}) as well as (5), provided the summation in (5) is carried out over all (l,m,n). Note that A_{lmn} is a quadratic function of a_{lmn}. If there are two terms a_{lmn} and a_{pqr} in u, then there are terms with the following (l,m,n) values in A_{lmn}:

$$(|l\pm p|,|m\pm q|,|n\pm r|),(|l\pm q|,|m\pm r|,|n\pm p|),$$
$$(l\pm r|,|m\pm p|,|n\pm q|), \tag{6}$$

and their permutations. By (5), we obtain $p_{lmn}=p_{mnl}=p_{nlm}=(-1)^l p_{lnm}$. From (1), we obtain the time evolution equation of a_{lmn},

$$\dot{a}_{lmn}+A_{lmn}-lp_{lmn}=0, \tag{7}$$

where the overdot denotes time derivative. Equation (7) and the mode-generation scheme (6) provide a prescription for the dynamical excitation of modes with increasingly large wave numbers, or a cascade of energy to small scales. If this process happens fast enough, then there may be a finite-time singularity.

Since the Euler equation preserves the Kida symmetries for all time, the selection rules imposed on a_{lmn}, A_{lmn}, and p_{lmn} by these symmetries are preserved by Eq. (7). By (5) and (7), we obtain the useful relation

$$p_{xx}\equiv-\sum_{lmn} l^2 p_{lmn}=0, \tag{8}$$

where p_{xx} denotes the second spatial derivative of p at the origin. It follows that $\nabla^2 p=0$ at the origin.

Let us now consider the flow along the line $y=z=0$. By (2), we obtain $v_y=v_z=0$, and $v_x=u=\Sigma_{lmn}a_{lmn}\sin lx$. Note that the vorticity is also identically zero along this line. We define $\alpha\equiv\partial_x v_x$, $\beta\equiv\partial_y v_y$, and $\gamma\equiv\partial_z v_z$, where $\alpha+\beta+\gamma=0$ by the incompressibility condition. The dynamical equation for α can be found by taking the x derivative of (1). We obtain

$$\dot{\alpha}+\alpha^2=-p_{xx}=\sum_{lmn} l^2 p_{lmn}\cos lx(t). \tag{9}$$

Equations similar to (9) can be written for β and γ. From (9), we obtain the following sufficient condition for a finite-time singularity: *If $p_{xx}>0$ for all time following a Lagrangian element, then α will be singular in finite time*. However, in order to test this condition, we need to evaluate p_{xx} by following a fluid element. It may be possible to test this condition numerically, but it is not convenient to do so analytically. We, therefore, attempt to obtain an Eulerian sufficient condition that can be evaluated at a fixed point (the origin).

Using (1), we write

$$\dot{v}_x=-p_x=\sum_{lmn} lp_{lmn}\sin lx(t), \tag{10}$$

where the overdot denotes total time derivative along a fluid element moving in a trajectory $x=x(t)$. By (8), for x close to zero, we obtain the Taylor expansion

$$\dot{v}_x=-\frac{p_{xxxx}}{6}x^3+O(x^5), \tag{11}$$

where p_{xxxx} is the fourth-order spatial derivative at the origin, given by $p_{xxxx}=\Sigma_{lmn}l^4 p_{lmn}$. Note that at the origin, although $\nabla^2\nabla^2 p=3p_{xxxx}+6p_{xxyy}=0$, p_{xxxx} is nonzero in general. For further reference, note also that $p_{xxxx}(x)$ is a symmetric function of x. The fourth-order derivative p_{xxxx} plays an important role because all other lower-order derivatives vanish at the origin in this highly symmetric flow. From the exact equation (9), we obtain

$$\dot{\alpha}+\alpha^2=-\frac{p_{xxxx}}{2}x^2(t)+O(x^4). \tag{12}$$

It is easy to see, using the selection rules for p_{lmn}, that $p_{xx}(x)$ is symmetric and $p_x(x)$ is antisymetric about $x=\pi/2$. Also, since $p_x=0$ at $x=0$ and $x=\pi/2$, we infer that p_{xx} has to assume both positive and negative values within the range $0<x<\pi/2$. Hence, there must be a region of x in which amplification of α^2 occurs. This leads to the next question: is there always a fluid element in the region of amplification? To answer this question, let us assume that there exists a range $0<x<X(t)$ in which $p_{xxx}(x)>0$ for all time before a possible singularity appears. Furthermore, we assume that $X(t)>C$, where C is a finite positive constant for all time (including $t\rightarrow t_c$). Since $p_{xxxx}(0)=p_x(0)=0$ by the symmetry conditions and $p_{xx}(0)=0$ by (8), it follows from the assumption above and by simple integration from $x=0$ that the quantities $p_x(x)$, $p_{xx}(x)$, and $p_{xxx}(x)$ are also positive within the range $0<x<X$. Then by (10) and the fact that $v_x(x,t=0)=0$, there exists a fluid element with the Lagrangian coordinate $x(t)$ within this range $(0,C)$ always accelerating towards the origin $x=0$. However, since the condition $v_x(x=0,t)=0$ is always maintained by the symmetry of the flow, the fluid element cannot pass through the origin.

Therefore, $x(t)$ always decreases but remains positive, even when the velocity becomes very large (or even singular). This seeming contradiction between what the trajectory tends to do and what it is constrained to do by symmetry is precisely the mechanism for the development of the finite-time singularity. On the one hand, $x(t)$ is always accelerated towards the origin and tends to reach the origin in finite time. On the other hand, $x(t)$ cannot actually reach the origin because the symmetry conditions forbid it. The system resolves this contradiction by having the velocity derivative $\alpha = \partial v_x / \partial x$ blow up in finite time, since the fluid element with finite and increasing velocity is forced to go infinitesimally close to the point with zero velocity $(x=0)$. This behavior is reflected in Eq. (9) according to which α tends to negative infinity in finite time due to the presence of the α^2 term. If the time dependence of α is determined dominantly by the α^2 term, then $\alpha \to (t_c - t)^{-1}$ as $t \to t_c$. Under the assumptions discussed above, we have thus demonstrated that the condition $p_{xxxx} > 0$ at the origin is a sufficient condition for a finite-time singularity.

We caution that the assumption of existence of a finite $X(t)$ is a strong one, and may limit the applicability of our sufficient condition at a fixed point. In some physical cases, as the singularity develops, $X(t)$ may actually tend to zero as $t \to t_c$. If that occurs and a fluid element falls out of the amplification region, t_c may tend to infinity. However, we speculate that if a shrinking $X(t)$ is accompanied by p_{xxxx} growing sufficiently fast, then the finite-time singularity may be supported because a fluid element is then accelerated fast enough to remain in $X(t)$ even as $X(t) \to 0$.

The sufficient condition for singularity (in its moving-point or fixed-point form) does not violate the theorems proved by Beale-Kato-Majda [1] and Ponce [2] which can be essentially summarized as follows: If there occurs a finite-time singularity in an initially smooth Euler flow of finite energy, then the time integral of the maximum norm of the vorticity [1] (deformation tensor [2]) must tend to infinity as $t \to t_c$. We remark that our discussion of the sufficient condition involves the $y=z=0$ axis on which the vorticity is identically zero by symmetry, but there is no restriction on the vorticity off the axis. If the sufficient condition is satisfied, the deformation tensor must be singular near the origin at least as fast as $1/(t_c - t)$, but this leaves open the possibility that the vorticity can blow up at another spatial location. Indeed, BP [9] report that the locations of near-divergent strain and vorticity do not coincide in space. Such a possibility is not inconsistent with [1] and [2]. BP [10] have recently checked that sufficient condition in their numerical simulation for the initial condition in one of their runs (run D3), and their data indicates that p_{xxxx} does remain positive and growing for all times from $t=0$ to the singularity time. (In particular, p_{xxxx} is found to take the following sequence of values: 40.53 at $t=0$, 226.4 at $t=1.5$, 2.642×10^4 at $t=2.0$, 2.802×10^5 at $t=2.125$, and 1.111×10^7 at $t=2.25$. The extrapolated singularity time t_c reported by BP for this run is 2.21.)

Though the numerical evidence presented above is suggestive, it cannot be regarded as definitive proof of the existence of the singularity. Furthermore, we cannot deduce generic properties of Kida flows from the numerical evidence for one specific initial condition. We now proceed to give a statistical demonstration that the positivity of p_{xxxx} is highly probable over a large ensemble of initial conditions.

First, we introduce a minimal, independent set of modes u_n of the Kida flow such that any flow u satisfying the symmetries of the Kida flow can be written as $u = \Sigma_n a_n u_n$, where a_n are real constants. Then, the following is the only possible choice for the set of independent modes that yield a minimum number of terms for each mode and satisfy the symmetries (4) and (5). For any three distinct, odd positive integers l,m,n, there are, in general, two independent modes:

$$\begin{aligned} u_1: &\quad a_{lmn} = -a_{lnm} = m, \quad a_{mnl} = -a_{mln} = -l, \\ u_2: &\quad a_{lmn} = -a_{lnm} = n, \quad a_{nlm} = -a_{nml} = -l. \end{aligned} \tag{13}$$

By (5), the third mode (u_3: $a_{mnl} = -a_{mln} = -n$, $a_{nlm} = -a_{nml} = m$) is not independent and can be written $u_3 = (nu_1 - mu_2)/l$. In practice, we can choose any two of the above set of three as independent. However, if two of the three integers are equal, there is only one independent mode, $a_{lml} = -a_{llm} = 1$. Also, there is no mode for $l=m=n$. The initial condition used by BP, that is, u_0: $a_{1,3,1} = 1$, $a_{1,1,3} = -1$, is clearly an independent mode. For any three even natural numbers, there are also, in general, only two independent modes:

$$\begin{aligned} u_1: &\quad a_{lmn} = a_{lnm} = m, \quad a_{mnl} = a_{mln} = -l, \\ u_2: &\quad a_{lmn} = a_{lnm} = n, \quad a_{nlm} = a_{nml} = -l. \end{aligned} \tag{14}$$

The third mode, which is not independent, is again $u_3 = (nu_1 - mu_2)/l$. If two of the three numbers are equal or one of them is zero, then there is only one independent mode. Also, there is no mode if two numbers are zero or one number is zero and the other two equal, or three numbers are equal.

The quantity p_{xxxx} (at the origin) is a quadratic function of u. We write $p_{xxxx} = P(u,u)$. With the representation of u, we obtain

$$p_{xxxx} = \sum_n a_n^2 P(u_n, u_n) + \sum_n \sum_{m \neq n} a_n a_m P(u_n, u_m). \tag{15}$$

Defining $P_{nm} = [P(u_n, u_m) + P(u_m, u_n)]/2$, we can also write $p_{xxxx} = \Sigma_n \Sigma_m a_n P_{nm} a_m$. We have a constructive proof using MATHEMATICA that $P(u_n, u_n) > 0$. The symbolic manipulations are too long to be presented here and will be discussed in a separate publication. In order to determine the positivity of p_{xxxx}, we now need to consider the second term on the right-hand side of (15), which involves the cross terms. The contributions from the cross terms cannot be neglected in principle, and they can be positive or negative depending on the sign of a_n. To assess their importance, let us consider an example with two modes, i.e., $u = a_m u_m + a_n u_n$, so that

$$P(u,u) = a_m^2 P_{mm} + a_n^2 P_{nn} + 2 a_m a_n P_{mn}. \tag{16}$$

The cross term P_{nm} is given by the relation $2P_{nm} = P(u_n + u_m, u_n + u_m) - P_{nn} - P_{mm}$. Note that the cross term between an odd mode and an even mode is always zero due to

TABLE I. Some results of the Monte Carlo calculations of p_{xxxx} with $\nu=3$ showing that $p_{xxxx}>0$ in most cases. See text and Eqs. (27) and (28) for definitions of the variables.

N	k_N^2	M (units of 10^6)	η (%)	$\langle P\rangle$	$\langle P_D\rangle$	$\langle d\rangle$	$\langle d_D\rangle$	$\langle r\rangle$
11	44	2	1.09	20.4	20.4	21.9	8.6	1.00
37	99	2	0.96	41.9	41.9	45.5	11	1.00
64	136	3	0.82	61.8	61.2	67.6	13	1.01
136	219	3	0.77	98.2	98.0	108	14	1.01
211	296	4	0.70	130	131	141	16	1.00
290	360	7	0.64	161	162	174	17	1.00
449	480	10	0.64	212	214	230	18	0.99
612	587	20	0.61	267	264	290	19	1.01
777	691	30	0.61	307	308	334	20	1.00
945	780	40	0.61	349	351	377	21	0.99
1114	875	50	0.58	394	390	423	22	1.01
1283	963	60	0.56	429	430	464	22	1.00
1455	1043	80	0.58	470	467	505	23	1.01
1624	1123	100	0.59	500	502	537	23	1.00
1973	1275	160	0.54	573	571	616	24	1.00
2144	1352	200	0.54	605	602	675	25	1.01

the selection rules on the pressure. For the positivity of $P(u,u)$ in (16), it is sufficient to have $P_{mm}P_{nn}\geqslant P_{nm}^2$. This relation is found to be true numerically if either u_m or u_n is the initial flow u_0 defined above, or the mode with $a_{3,3,1}=1$ and $a_{3,1,3}=-1$. However, the condition $P_{mm}P_{nn}\geqslant P_{nm}^2$ is not always true for any two modes. Instead, we have shown by using MATHEMATICA that

$$\frac{P_{nm}^2}{P_{mn}P_{nn}}\rightarrow 0 \quad \text{as} \quad \frac{k_m^2}{k_n^2}\rightarrow 0, \tag{17}$$

where k_m, k_n are the wave numbers of the modes u_m, u_n. The limit (17) is attained faster by odd modes than by the even modes. Hence, for those cases, which make up most of the pairs of modes for a flow spanning wave numbers over several orders of magnitude, the contribution from off-diagonal terms is much smaller than that from diagonal terms.

To estimate the contributions from the cross terms systematically in a general case, let us consider a flow represented by N modes ($n=1,N$) that includes all modes with $k_n<k_N$. For normalization, we define an energy functional

$$E(u)\equiv\frac{1}{\pi^3}\int_0^{2\pi}dx\int_0^{2\pi}dy\int_0^{2\pi}dz u^2=2\sum_{lm}a_{lm0}^2+\sum_{lmn}a_{lmn}^2, \tag{18}$$

which is conserved in an Euler flow. For the specific initial condition used in [7] and [8], we obtain $E(u_0)=2$. Hence, we normalize the u_n in (18) such that $E(u_n)=2$. We now perform a Monte Carlo calculation of $P(u,u)$. The calculation is carried out M times with $M\gg 1$, with a_n chosen randomly each time within a range specified by an energy spectrum in k space of the form $E(k)\propto k^{-\mu}$ to simulate the energy distribution over different length scales. In most of our calculations, we choose $\mu=3$ which is the spectrum observed by BP near $t=t_c$ [7]. (As discussed later, the qualitative trends observed are not sensitive to variations in μ.) In Table I, we report numerical results with $N\leqslant 2144$, or $k_N^2\leqslant 1352$. We define η as the percentage of cases with negative vales for p_{xxxx}, and the averaged quantities are

$$\langle P\rangle\equiv\frac{1}{M}\sum_{i=1}^{M}p_{xxxx},$$
$$\langle P_D\rangle\equiv\frac{1}{M}\sum_{i=1}^{M}\sum_{n=1}^{N}a_n^2P_{nn}=\frac{1}{M}\sum_{i=1}^{M}P_D, \tag{19}$$

where P_D is the contribution from the diagonal terms. Also, we define the deviations and the average ratio as

$$\langle d\rangle\equiv\left[\frac{1}{M}\sum_{i=1}^{M}(p_{xxxx}-\langle P\rangle)^2\right]^{1/2},$$
$$\langle d_D\rangle\equiv\left[\frac{1}{M}\sum_{i=1}^{M}(P_D-\langle P_D\rangle)^2\right]^{1/2}, \tag{20}$$
$$\langle r\rangle\equiv\frac{1}{M}\sum_{i=1}^{M}\frac{p_{xxxx}}{P_D}.$$

The convergence of the data is tested for large M in two cases. The fluctuations in all quantities are found to be typically less than 2%.

We examine the sensitivity of the results to the assumed form of the energy spectrum by recalculating the $N=177$ case with $\mu=-2,0,2,4,6$. It is found for these cases that η becomes 1.4%, 1.31%, 0.91%, 0.35%, 0.26%, respectively. Hence, the effect of μ is seen not to be qualitatively important. From the data in Table I, we see that η is less than 1% for most cases, except the cases with very small N. If this trends continues to hold for larger N values, then the probability that $p_{xxxx}>0$ is much larger than the probability that $p_{xxxx}<0$. Note also that for all the cases discussed above, $\langle P\rangle\approx\langle P_D\rangle\approx\langle d\rangle$, $\langle r\rangle\approx 1$. This implies that the average contribution to p_{xxxx} is dominantly from the self terms, and that the cross terms mostly cancel each other upon summation. We also see from Table I that $\langle P\rangle \propto k_N^2$ which, in light of our remarks above, is further evidence in support of a finite-time singularity because the data clearly shows the growth of p_{xxxx} as length scales decrease.

The numerical results presented above are based mostly on statistics. Since the deterministic dynamics of Kida flows do not have to follow the most probable path, we cannot regard the evidence above as a dynamical proof that $p_{xxxx}>0$, However, the evidence does suggest that the condition $p_{xxxx}>0$ is highly probable and is strongly favored by the symmetry properties of the Kida flow, independent of the precise dynamical details emerging from a specific initial condition.

The high symmetry of the Kida flow enables us to obtain some analytical and numerical results that provide strong physical evidence in support of a finite-time singularity in this class of Euler flows. The assumed symmetry properties preserve the geometric structure of the initial state for all times. Such a singularity may be unstable if the symmetry conditions are relaxed, and so the qualitative implications of these results for more general 3D configurations remain unclear. Till 1990, numerical results on 3D flows were inconclusive despite the sophistication of the numerical methods employed [11–15]. Finite-time singularities have been reported in axisymmetric flows with swirl [16–18], but the results are controversial [19,20]. More recently, prior to the work of BP, two other numerical experiments [21,22] have presented evidence in support of a finite-time singularity in 3D Euler flows. In particular, Kerr's simulation [22] involves antiparallel vortex tubes, and has qualitatively similarities with that of BP in that the singularity occurs in the vicinity of the symmetry axis. The work of Kerr has its antecendents in earlier studies of vortex reconnection with antiparallel and orthogonal vortex tubes [23–29].

In conclusion, we have proposed a sufficient condition for a finite-time singularity in a Kida flow. We have shown that if the second-order spatial derivative of the pressure (p_{xx}) is positive following a Lagrangian element (on the x axis), then a finite-time singularity must occur. Under some assumptions, this Lagrangian sufficient condition can be reduced to an Eulerian sufficient condition which requires that the fourth-order spatial derivative of the pressure (p_{xxxx}) at the origin be positive for all times leading up to the singularity. Though we are unable to provide an analytical proof that this is indeed satisfied in the simulation of BP [7], there is numerical evidence [10] in support of this condition. Furthermore, we have presented strong physical evidence which suggests that it is highly probable that the Eulieran form of the sufficient condition for singularity is satisfied for a large ensemble of initial conditions.

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